FUNCTIONAL ANALYSIS

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Basic informations

The only evaluation is the final exam.

Exam requirements:

- Orientation in the topics
- Understanding relationships of different notions
- Knowledge of proofs of all theorems

Syllabus

Topological Spaces

Let X be a non-empty set. A *topology* on X is a family $\mathcal{T}$ of subsets of X, which are called *open* sets, with the following properties:

(1) both the empty set $\emptyset$ and the total set X are open,
(2) an arbitrary union of open sets is open, and
(3) a finite intersection of open sets is open.

The system $(X, \mathcal{T})$ is called a *topological space*.

A subset F of X is said to be *closed* if and only if its complement $X - F$ is open.

We say that $f : X \to Y$ is *continuous* if $f^{-1}(G)$ is open for every G open in Y.

A collection $\mathcal{B}$ of open sets is said to be a base for the topology $\mathcal{T}$ on a space X if and only if each non-empty element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $X_1 \times X_2$ be their product. Sets of the form $U \times V$, with $U \in \mathcal{T}_1$ and $V \in \mathcal{T}_2$, form a base for the *product topology*.

A *pseudometric* on a space X is a function $d : X^2 \to [0, \infty)$ satisfying the following axioms:

(1) $d(x, x) = 0$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leqq d(x, y) + d(y, z)$.

It is a metric if it satisfies the additional axiom:

(4) $d(x, y) = 0 \Rightarrow x = y$.

Let X be a metric space. The *open ball* of radius $\varepsilon > 0$ centered at a point $p \in X$ is defined as

$$B(p, \varepsilon) = \{x \in X : d(x, p) < \varepsilon\}.$$

A sequence $(x_n)_{n \in \mathbb{N}}$ is *Cauchy* if and only if

$$\forall \varepsilon > 0 \ \exists n \in \mathbb{N} \ \forall k, l \geqq n : d(x_k, x_l) < \varepsilon.$$

A metric space X is *complete* if and only if every Cauchy sequence in X converges.

Linear Spaces

A set X is called a *real* (*complex*) *linear space* if for every two of its elements $\boldsymbol{x}$ and $\boldsymbol{y}$ there is assigned an element $\boldsymbol{x}+\boldsymbol{y}$ of the set, called their *sum*, and if for any real (complex) number λ and any element $\boldsymbol{x}$ there is assigned an element $\lambda\boldsymbol{x}$ of the set X, called their *product*, where these operations satisfy the following conditions:

(1) $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$,
(2) $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$,
(3) there exists an element Θ in X such that $0\boldsymbol{x}=\Theta$ for every $\boldsymbol{x}\in X$,
(4) $(\lambda+\mu)\boldsymbol{x}=\lambda\boldsymbol{x}+\mu\boldsymbol{x}$,
(5) $\lambda(\boldsymbol{x}+\boldsymbol{y})=\lambda\boldsymbol{x}+\lambda\boldsymbol{y}$,
(6) $(\lambda\mu)\boldsymbol{x}=\lambda(\mu\boldsymbol{x})$,
(7) $1\boldsymbol{x}=\boldsymbol{x}$.

A finite set of elements $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$ in a linear space X over $\mathbb{K}$ is said to be *linearly independent* if and only if $\alpha_1\boldsymbol{x}_1+\dots\alpha_n\boldsymbol{x}_n=\Theta$ with $\alpha_1,\dots,\alpha_n\in\mathbb{K}$ implies that $\alpha_1=\dots=\alpha_n=0$. A subset A in a linear space is said to be *linearly independent* if and only if each finite subset of A is.

A subset H in a linear space X is called a *Hamel basis* of X if and only if it is linearly independent and if any element of X can be written as a finite combination of elements of H.

Theorem. *Let A be a linearly independent subset of a linear space X. Then there is a Hamel basis of X containing A.*

Let X be a linear space over a field $\mathbb{K}$, and let A be a subspace of X. We define an equivalence relation $\sim$ on X by standing that $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} - \boldsymbol{y} \in A$. The equivalence class of $\boldsymbol{x}$ is $\boldsymbol{x} + A$. The quotient space X/A is defined as $X/\sim$, the set of all equivalence classes over X by $\sim$. Scalar multiplication and addition are defined on the equivalence classes by

(1) $\alpha(\boldsymbol{x} + A) = (\alpha\boldsymbol{x}) + A$ for all $\alpha \in \mathbb{K}$, and

(2) $(\boldsymbol{x} + A) + (\boldsymbol{y} + A) = (\boldsymbol{x} + \boldsymbol{y}) + A$.

Topological Linear Spaces

The field $\mathbb{K}$ of scalars will always be $\mathbb{R}$ or $\mathbb{C}$. A *topological linear space* over $\mathbb{K}$ is a linear space X over $\mathbb{K}$ furnished with a topology $\mathcal{T}$ such that

(1) the map $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} + \boldsymbol{y}$ is continuous from X^2 into X (where X^2 is given the product topology),

(2) the map $(\lambda, \boldsymbol{x}) \mapsto \lambda\boldsymbol{x}$ is continuous from $\mathbb{K} \times X$ into X (where $\mathbb{K}$ has its usual topology, and $\mathbb{K} \times X$ the product topology).

Topological linear spaces are uniform spaces.

A set A is called *absorbing* if for all $\boldsymbol{x} \in X$ there exists a real number r such that $\forall \alpha \in \mathbb{K} : |\alpha| \geqq r \Rightarrow \boldsymbol{x} \in \alpha A$ with $\alpha A = \{\alpha\boldsymbol{x} : \boldsymbol{x} \in A\}$.

A set A is called *balanced* if for all $\alpha \in \mathbb{K}$ with $|\alpha| \leqq 1$ we have $\alpha A \subset A$.

Theorem. *Every finite dimensional Hausdorff topological linear space X over $\mathbb{K}$ with dimension n is linearly homeomorphic to $\mathbb{K}^n$.*

Normed Spaces

Let X be a linear space over $\mathbb{K}$. A mapping $\|\cdot\| : X \to \mathbb{R}$ is said to be a *seminorm* if

(1) $\|\boldsymbol{x}\| \geqq 0$ for all $\boldsymbol{x} \in X$,
(2) $\|\boldsymbol{x} + \boldsymbol{y}\| \leqq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in X$,
(3) $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in X$ and all $\lambda \in \mathbb{K}$.

If $\|\cdot\|$ has the additional property that $\|\boldsymbol{x}\| = 0$ implies that $\boldsymbol{x} = \Theta$, then $\|\cdot\|$ is a *norm* on X. A *normed space* is a linear space possessing a norm.

If X is a normed space with the norm $\|\cdot\|$, then the formula $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$, for $\boldsymbol{x}, \boldsymbol{y} \in X$, defines a metric d on X. This is called the *metric induced by the norm* $\|\cdot\|$.

Theorem. *Any two norms on a finite dimensional linear space are equivalent.*

Locally Convex Spaces

A subset A of a linear space X over $\mathbb{K}$ is said to be *convex* if $\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}\in A$ whenever $\boldsymbol{x},\boldsymbol{y}\in A$ and $0\leqq\lambda\leqq 1$.

A topological linear space is said to be *locally convex* if there is a neighborhood base at Θ consisting of convex sets.

Theorem. *For any convex absorbing set $V\subset X$ the function $p_V : X\to\mathbb{R}$, $p_V(\boldsymbol{x})=\inf\{t>0 : \boldsymbol{x}\in tV\}$, is positively homogeneous and subadditive on X. If, in addition, V is balanced, then p_V is a seminorm on X. Furthermore, $\{\boldsymbol{x}\in X : p_V(\boldsymbol{x})<1\}\subset V\subset\{\boldsymbol{x}\in X : p_V(\boldsymbol{x})\leqq 1\}$.*

The function p_V is called the *Minkowski functional* associated with the convex absorbing set V in the linear space X.

Theorem. *The topology of any locally convex topological linear space is determined by a family of seminorms. This family may be taken to be the family $\mathcal{P}$ of Minkowski functionals associated with all convex, absorbing, balanced neighborhoods of Θ.*

Banach Spaces

A complete normed space is called a *Banach space*. A normed space X is a Banach space if every Cauchy sequence in X converges.

Proposition. *The normed space X is complete if and only if the series $\sum_{n=1}^{\infty} \boldsymbol{x}_n$ converges, where $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is any sequence in X satisfying $\sum_{n=1}^{\infty} \|\boldsymbol{x}_n\| < \infty$. In other words, a normed space is complete if and only if every absolutely convergent series is convergent.*

Let A be a linear subspace of X. We define the *quotient norm* on the quotient space X/A by the following way

$$\|\boldsymbol{x} + A\| = \inf\{\|\boldsymbol{y}\| : \boldsymbol{y} \in \boldsymbol{x} + A\}.$$

Theorem. *For any closed linear subspace A of a Banach space X, the quotient space X/A is a Banach space under the quotient norm.*

Linear Operators

A *linear operator* T between linear spaces X and Y (over $\mathbb{K}$) is a map $T : X \to Y$ such that

$$T(\alpha\boldsymbol{x} + \beta\boldsymbol{y}) = \alpha T(\boldsymbol{x}) + \beta T(\boldsymbol{y})$$

for all $\alpha, \beta \in \mathbb{K}$ and all $\boldsymbol{x}, \boldsymbol{y} \in X$.

The linear operator $T : X \to Y$ is said to be *bounded* if there exists some $b > 0$ such that $\|T(\boldsymbol{x})\| \leqq b\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in X$. In this case we define $\|T\|$ to be

$$\|T\| = \inf\{b : \|T(\boldsymbol{x})\| \leqq b\|\boldsymbol{x}\| \text{ for all } \boldsymbol{x} \in X\}.$$

Proposition. *Suppose that $T : X \to Y$ is a bounded linear operator. Then*

$$\|T\| = \sup\{\|T(\boldsymbol{x})\| : \|\boldsymbol{x}\| \leqq 1\} = \sup\{\|T(\boldsymbol{x})\| : \|\boldsymbol{x}\| = 1\} = \sup\left\{\frac{\|T(\boldsymbol{x})\|}{\|\boldsymbol{x}\|} : \boldsymbol{x} \neq \Theta\right\}.$$

The set of bounded linear operators from a normed space X into a normed space Y is denoted $\mathcal{B}(X, Y)$.

The Dual Space

Let X be a linear space over $\mathbb{K}$. The *algebraic dual* of X (often denoted X^*) is the vector space over $\mathbb{K}$ of all linear functionals on X equipped with the obvious operations of addition and scalar multiplication.

Let X be a normed space over K. The space of all bounded linear functionals on X is denoted by X' and called the *continuous dual* of X. When the context is clear, the continuous dual may just be called the dual.

Theorem. *Let X be a normed space. For $x \in X$, let $V_{\boldsymbol{x}} : X' \to \mathbb{K}$ be the evaluation map $V_{\boldsymbol{x}}(f) = f(\boldsymbol{x})$, $f \in X'$. Then $\boldsymbol{x} \mapsto V_{\boldsymbol{x}}$ is an isometric linear mapping of X into X''.*

Thus we may consider X as a subspace of X'' via the linear isometric embedding $\boldsymbol{x} \mapsto V_{\boldsymbol{x}}$. A Banach space X is called *reflexive* if $X = X''$ via the above embedding.

Theorem. *A Banach space X is reflexive if and only if X' is reflexive.*

Baire Category Theorem

A topological space is called a *Baire space* if the countable union of any collection of closed sets with empty interior has empty interior.

Baire Category Theorem. *Every non-empty complete metric space is a Baire space.*

Theorem. *(Banach-Steinhaus principle of uniform boundedness) Let X be a Banach space and let $\mathcal{F}$ be a family of bounded linear operators from X into a normed space Y such that for each $\boldsymbol{x} \in X$ the set $\{\|T(\boldsymbol{x})\| : T \in \mathcal{F}\}$ is bounded. Then the set of norms $\{\|T\| : T \in \mathcal{F}\}$ is bounded.*

Theorem. *(Open mapping theorem) Suppose that both X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator. Then T is an open map, i. e. T maps open sets in X into open sets in Y.*

Theorem. *(Closed graph theorem) Suppose that X and Y are Banach spaces and $T : X \to Y$ is a linear operator. Then T is bounded if and only if the graph of T is closed.*

Hahn-Banach Theorem

Hahn-Banach Theorem. *(real version) Let X be a real linear space and suppose that $f : M \to \mathbb{R}$ is a linear mapping defined on a linear subspace M of X such that $f(y) \leqq p(y)$, for all $y \in M$, for some subadditive and positively homogeneous map $p : X \to \mathbb{R}$. Then there is a linear functional $L : X \to \mathbb{R}$ such that $L(x) = f(x)$ for $x \in M$ and*

$$-p(-x) \leqq L(x) \leqq p(x) \text{ for } x \in X.$$

Hahn-Banach Theorem. *(complex version) Suppose that M is a linear subspace of a vector space X over $\mathbb{C}$, p is a seminorm on X, and f is a linear functional on M such that*

$$|f(x)| \leqq p(x) \text{ for } x \in M.$$

Then there is a linear functional L on X such that $L(x) = f(x)$ for $x \in M$ and

$$|L(x)| \leqq p(x) \text{ for all } x \in X.$$

L^p SPACES

Let $(X, \mathcal{B}, \mu)$ be a measurable space. Let $0 < p < \infty$. The L^p-norm of a function $f : X \to \mathbb{C}$ is defined as

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}$$

when the integral exists. The set of functions with finite L^p-norm forms a linear space V with the usual pointwise addition and scalar multiplication of functions. The set K of functions with zero L^p-norm form a linear subspace of V.

Theorem. *If $1 \leqq p < \infty$, the quotient space V/K is complete with respect to the L^p-norm.*

The L^∞-norm of f is defined as follows:

$$\|f\|_\infty = \inf\{a \in \mathbb{R} : \mu(\{x : |f(x)| > a\}) = 0\}.$$

ℓ^p spaces are L^p spaces for the counting measure on $\mathbb{N}$.

Hilbert Spaces

Let X be a complex linear space. A complex-valued function

$$\langle\cdot,\cdot\rangle : X^2 \to \mathbb{C}$$

of two variables on X is a *inner product* if
(1) $\langle x,x\rangle \geqq 0$ and equality only for $x = \Theta$,
(2) $\langle x,y\rangle = \overline{\langle y,x\rangle}$,
(3) $\langle x,y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$,
(4) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$.

Then X equipped with such a $\langle\cdot,\cdot\rangle$ is a *pre-Hilbert space.*

The associated norm $\|\cdot\|$ on X is defined by

$$\|x\| = \sqrt{\langle x,x\rangle}.$$

If a pre-Hilbert space is complete with respect to the metric arising from its inner product, then it is called a *Hilbert space.*

Theorem. *(Riesz representation) If T is a bounded linear functional on a Hilbert space X, then there exists a unique vector $\boldsymbol{y} \in X$ such that $T(\boldsymbol{x}) = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$ for all $\boldsymbol{x} \in X$.*

An important consequence of the Riesz representation theorem is the existence of the *adjoint* of a bounded on a Hilbert space. The defining property of the adjoint $T^* \in \mathcal{B}(X)$ of an operator $T \in \mathcal{B}(X)$ is that $\langle \boldsymbol{x}, T(\boldsymbol{y}) \rangle = \langle T^*(\boldsymbol{x}), \boldsymbol{y} \rangle$ for all $\boldsymbol{x}, \boldsymbol{y} \in X$.

A bounded linear operator $T : X \to X$ on a Hilbert space X is *self-adjoint* if $T^* = T$. Equivalently, a bounded linear operator T is self-adjoint if and only if $\langle \boldsymbol{x}, T(\boldsymbol{y}) \rangle = \langle T(\boldsymbol{x}), \boldsymbol{y} \rangle$ for all $\boldsymbol{x}, \boldsymbol{y} \in X$.

The Spectrum of Bounded Linear Operators

We denote the space of bounded linear operators on a Hilbert space X by $\mathcal{B}(X)$. The *resolvent set* of an operator $T \in \mathcal{B}(X)$, denoted by $\rho(T)$, is the set of complex numbers λ such that $(T - \lambda I) : X \to X$, where $I : X \to X$ is the identity map, is one-to-one and onto. The *spectrum* of T, denoted by $\sigma(T)$, is the complement of the resolvent set in $\mathbb{C}$. A complex number λ is called an *eigenvalue* of T if there is a non-zero vector $\boldsymbol{x} \in X$ such that $T(\boldsymbol{x}) = \lambda \boldsymbol{x}$.

The *spectral radius* of T, denoted by $r(T)$, is the radius of the smallest disk centered at zero that contains $\sigma(T)$, i. e. $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

Theorem. *(Spectral theorem for compact, self-adjoint operators) Let $T : X \to X$ be a compact, self-adjoint operator on a Hilbert space X. There is an orthonormal basis of X consisting of eigenvectors of T. The non-zero eigenvalues of T form a finite or countably infinite set $\{\lambda_k\}$ of real numbers, and $T = \sum_k \lambda_k P_k$, where P_k is the orthogonal projection onto the finite-dimensional eigenspace of eigenvectors with eigenvalue λ_k.*

Literature

E. Hille, R. S. Phillips: *Functional Analysis and Semi-groups*, ISBN 0-8218-3395-2.

I. F. Wilde: *Functional Analysis – Notes.*

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