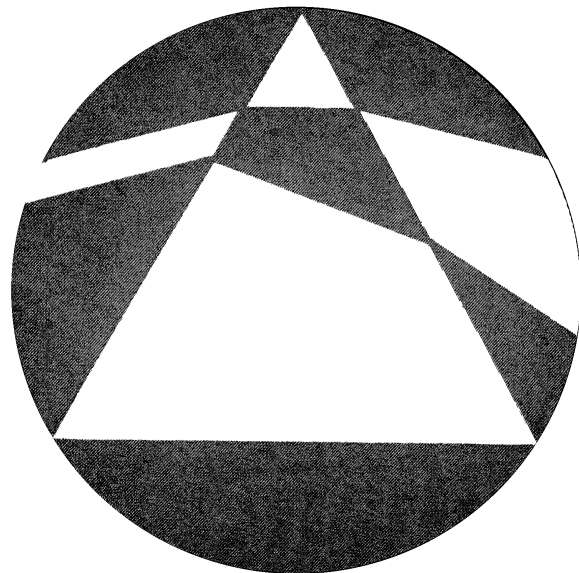


# Mathematical Spectrum

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**A magazine for students and teachers of mathematics  
in schools, colleges and universities,  
and for everyone interested in mathematics**



**Volume 41   2008/2009   Number 1**

- Sudoku Grids
- Henges, Heel Stones, and Analemmas
- Mathematics and Music

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One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing. The copyright of all published material is vested in the Applied Probability Trust.

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# Some Irrationals do not have Special Names

JOZEF DOBOŠ

The proof that  $\sqrt{2}$  is irrational appears in textbooks as a standard introduction to proof by contradiction. Numbers like  $\sqrt{2}$  fall into a special category of irrational numbers of the form  $\sqrt[n]{a}$ , where  $a$  is not a perfect  $n$ th power. These numbers can be seen as the solution to  $x^n = a$ . Other irrational numbers can be easily obtained as solutions to polynomial equations. For example,  $\sqrt{2} + \sqrt{3}$  and  $2 \cos(\pi/7)$  are solutions to  $x^4 - 10x^2 + 1 = 0$  and  $x^3 + x^2 - 2x - 1 = 0$  respectively. The irrationality of these numbers follows from the rational root theorem (see reference 1). Yet more irrational numbers arise when you solve nonalgebraic equations. Usually they do not have special names. The aim of this article is to present an elementary approach to irrationality of roots of some charming equations of this type.

**Example 1** (See reference 2.) Evidently,  $x = 3$  is a solution to

$$3^x = x^3. \quad (1)$$

But there is another real number that satisfies this equation and is irrational.

Its existence follows from Bolzano's theorem, which says that a continuous function cannot change its sign without going through the zero. (This result is also known as the mean value theorem.) Put

$$f(x) = 3^x - x^3.$$

Since  $f(2) > 0$  while  $f(\frac{5}{2}) < 0$ , and  $f$  is a continuous function, there must be some number  $x$  between 2 and  $\frac{5}{2}$  with the property  $f(x) = 0$ .

Since for each real number  $x$  we have  $3^x > 0$ , (1) is equivalent to

$$\frac{\ln x}{x} = \frac{\ln 3}{3}.$$

Since

$$\left(\frac{\ln x}{x}\right)' = \frac{1 - \ln x}{x^2},$$

the function  $\ln x/x$  is increasing on the interval  $(0, e)$  and it is decreasing for  $x > e$ .

Therefore, (1) has exactly one root in the interval  $(0, e)$ , denoted by  $z$ , say, and it has exactly one root larger than  $e$ , namely  $x = 3$ , as we know.

Now we show that  $z$  is irrational by contradiction. Assume that

$$z = \frac{p}{q},$$

where  $p$  and  $q$  are positive integers and  $p/q$  is in lowest terms. Substituting  $p/q$  for  $x$  into (1) gives

$$\begin{aligned} 3^{p/q} &= \left(\frac{p}{q}\right)^3, \\ 3^p &= \left(\frac{p}{q}\right)^{3q}, \\ 3^p q^{3q} &= p^{3q}. \end{aligned} \quad (2)$$

Hence,  $p^{3q}$  is divisible by 3. This means that  $p$  must be as well. Substituting  $p = 3r$ , where  $r$  is a positive integer, into (2) gives

$$\begin{aligned} 3^{3r} q^{3q} &= (3r)^{3q} \\ q^{3q} &= 3^{3(q-r)} r^{3q}. \end{aligned} \quad (3)$$

Since  $z < 3$ , we have  $r < q$ , which yields that the right-hand side of (3) is divisible by 3. So  $q^{3q}$  is divisible by 3. This means that  $q$  must be as well. But this also means that  $p$  and  $q$  are both divisible by 3, so  $p/q$  was not in lowest terms, which is a contradiction. The numerical value of  $z$  is 2.478 052 680 288 . . . .

**Example 2** (See references 3 and 4.) It is clear that  $x = \frac{1}{2}$  and  $x = \frac{1}{4}$  are solutions to

$$a^{a^x} = x, \quad (4)$$

where  $a = \frac{1}{16}$ . But there is another real number that satisfies this equation and is irrational.

Since  $a^x = x$  implies  $a^{a^x} = x$ , we start with the equation

$$a^x = x. \quad (5)$$

Put

$$f(x) = a^x - x.$$

Since  $f(\frac{1}{4}) = \frac{1}{4} > 0$  while  $f(\frac{1}{2}) = -\frac{1}{4} < 0$ , and  $f$  is a continuous function, there must be some number  $x$  between  $\frac{1}{4}$  and  $\frac{1}{2}$  with the property  $f(x) = 0$ . The reason for this number being unique is that the function  $f$  is decreasing (as a sum of two decreasing functions). Therefore, (5) has exactly one root, denoted by  $z$ , say.

Now we show by contradiction that  $z$  is irrational. Assume that

$$z = \frac{p}{q},$$

where  $p$  and  $q$  are positive integers and  $p/q$  is in lowest terms. Substituting  $p/q$  for  $x$  into (5) gives

$$a^{p/q} = \frac{p}{q},$$

or

$$q^q = 2^{4p} p^q. \quad (6)$$

So  $q^q$  is even. This means that  $q$  must be even as well. Substituting  $q = 2r$ , where  $r$  is a positive integer, into (6) gives

$$\begin{aligned}(2r)^{2r} &= 2^{4p} p^{2r}, \\ r^{2r} &= 2^{4p-2r} p^{2r}.\end{aligned}\tag{7}$$

Since  $z > \frac{1}{4}$ , we have  $4p > 2r$ , which yields that the right-hand side of (7) is even. So  $r^{2r}$  is even. This means that  $r$  must be even as well.

Substituting  $r = 2s$ , where  $s$  is a positive integer, into (7) gives

$$\begin{aligned}(2s)^{4s} &= 2^{4p-4s} p^{4s}, \\ 2^{8s-4p} s^{4s} &= p^{4s}.\end{aligned}\tag{8}$$

Since  $z < \frac{1}{2}$ , we have  $4p < 8s$ , which yields that the left-hand side of (8) is even. So  $p^{4s}$  is even. This means that  $p$  must be even as well.

So  $p$  and  $q$  are both even which is a contradiction to our assumption that they have no common factors. The numerical value of  $z$  is 0.364 249 889 784 . . . .

Finally, we show that (4) has no other roots. Since for each real number  $x$  we have  $a^x > 0$ , (4) is equivalent to

$$f(x) = 0,\tag{9}$$

where

$$f(x) = 16^x \ln x + \ln 16.$$

Then

$$f'(x) = \frac{16^x}{x} g(x),$$

where

$$g(x) = x \ln x \ln 16 + 1.$$

It is not difficult to verify that the function  $g(x)$  is decreasing on  $(0, 1/e)$  and increasing for  $x > 1/e$ . Since  $g(\frac{1}{4}) = g(\frac{1}{2}) > 0$  and  $g(1/e) < 0$ , there are positive real numbers  $a$  and  $b$  such that  $g(x) < 0$  for each  $x \in (a, b)$  and  $g(x) > 0$  for each  $x \in (0, a) \cup (b, \infty)$ . Therefore, the function  $f(x)$  is decreasing on the interval  $[a, b]$  and it is increasing on the intervals  $(0, a]$  and  $[b, \infty)$ . This yields that (9) can have at most three roots.

## References

- 1 <http://www.math.ou.edu/~jalbert/putnam/putnam9.pdf>
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ISSN 0025-5653

<http://www.appliedprobability.org>  
**Published by the Applied Probability Trust**  
Printed by MFK Pear Tree Press Ltd, Stevenage, Hertfordshire, UK